SEMESTER I ~~Unit 05~~ Unit-05

THEORY OF Equation

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5.1 Elementary Properties of Equations

5.1.1 Introduction :-

Theory of equation is the part of Mathematics in which we analysis the nature and algebraic solutions of polynomial equations.

5.1.2 Formation of Equations :-

Whenever variables are multiplied by some scalar (rational number) and combined with algebraic notations gives an expression which is equivalent to zero, called as equations.

Ex:- $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ is an equation of degree n with variable x.

We can form the following types of equations:-

① Algebraic equation ② Rational equations
③ Irrational equation ④ Modulus equation
⑤ Logarithmic equation ⑥ Exponential equation
⑦ Trigonometric equation.

5.1.3 Polynomial equations :-

The general form of a polynomial equation of nth degree is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants.

Note ① The highest power of x in a polynomial determines the degree of polynomial equations

② Equations are classified into following categories depending ~~on~~ upon degree

① Linear equation : $ax+b=0$; degree $=1$

(ii) Quadratic equations: $ax^2+bx+c=0$; degree $=2$
(iii) Cubic equation : $ax^3+bx^2+cx+d=0$; degree $=3$
or $a_0x^3+a_1x^2+a_2x+a_3=0$
(iv) Biquadratic eqn: $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$; degree $=4$
(v) Quintic equations ; degree $=5$
(vi) Sextic equations ; degree $=6$ etc.
(vii)

5.1.4 Division Algorithm for polynomials:-

For any two polynomials $f(x)\neq 0$ and $g(x)\neq 0$, there exists uniquely two polynomials $q(x)$ and $r(x)$ such that $f(x)=g(x)\cdot q(x)+r(x)$.
where either $r(x)=0$ or degree $r(x)<$ deg. $g(x)$.

① Remainder theorem :-

If we put $g(x)=(x-a)$ · (a polynomial of first degree)

Then,
$$f(x)=(x-a)\cdot q(x)+r(x)$$
$$=(x-a)\cdot q(x)+r \text{ , where } r \text{ is constant}$$
$$\therefore f(a)=(a-a)\cdot q(x)+r$$
$$\therefore r=f(a) \text{ , which is remainder.}$$

② Factor theorem:- If $f(a)=0$, then the remainder is zero.
i.e. the polynomial is completely divisible by $(x-a)$.

5.1.5 Theorem :- If $f_n(x)$ is divisible by $(x-\alpha)$, then α shall be a root of equation $f_n(x)=0$.

Conversely, if α be a root of the equation $f_n(x)=0$, then $f_n(x)$ shall be divisible by $(x-\alpha)$. Prove it.

Proof:-

Let Q denote the quotient and R the remainder when $f_n(x)$ is divided by $(x-\alpha)$.

$$\therefore \; x-\alpha)\, f_n(x)\, (\, Q$$
$$\overline{R}$$

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We have, $f_n(x) \equiv (x-\alpha)\cdot Q + R$ —— ①
If $f_n(x)$ is exactly divisible by $(x-\alpha)$. Then we must have $R=0$
So, Eq ① becomes, $f_n(x) \equiv (x-\alpha) Q$ —— ②
Now, if we put $x=\alpha$ in ②, we get $f_n(x)=0$
Hence, $x=\alpha$ is a root of equation $f_n(x)=0$.
Conversely,
If $x=\alpha$ is a root of eq^n $f_n(x)=0$
Then, we have $f_n(\alpha)=0$
Putting $x=\alpha$ in ①, we get

$$f_n(x) = (\alpha-\alpha)Q + R$$

$$\text{or,} \quad 0 = 0\cdot Q + R = R$$

i.e. Remainder is zero when $f_n(x)$ is divided by $(x-\alpha)$.

Proved.

[5.1.6]. Fundamental Theorem of Algebra:

Statement:- "Every polynomial equation with real coefficients has at least one root, real or complex."

[5.1.7] Theorem:- To Prove "Every equation of nth degree has n roots and no more"

Proof:- Suppose,

$$f_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad \text{—— ①}$$

is a polynomial of degree $n \geq 1$ and $a_0 \neq 0$

From fundamental theorem of Algebra, $f_n(x)$ has at least one root, say α_1.
Then, $(x-\alpha_1)$ is a root of $f_n(x)$.
So, $f_n(x) \equiv (x-\alpha_1)\cdot f_{n-1}(x)$ —— ②
where, $f_{n-1}(x)$ is a polynomial of $(n-1)^{th}$ degree.
According to fundamental theorem of Algebra, $f_{n-1}(x)=0$ has a root, say α_2

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Then, $f_{n-1}(x)$ is divisible by $(x-\alpha_2)$.

$\therefore f_{n-1}(x) \equiv (x-\alpha_2) f_{n-2}(x)$

and consequently from ②,

$$f_n(x) \equiv (x-\alpha_1)(x-\alpha_2) f_{n-2}(x) \quad \text{③}$$

Similarly, $f_{n-2}(x)=0$ has a root, say α_3

Then, $f_{n-2}(x) \equiv (x-\alpha_3)\cdot f_{n-3}(x)$

Thus, eq ③ becomes,

$$f_n(x) \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdot f_{n-3}(x)$$

continuing this process, we get

$$f_n(x) \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \cdots (x-\alpha_n)\, Q \quad \text{④}$$

Now, each of $f_n(x)$ and $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$ is of nth degree, hence Q is free from x.

Hence, equating the coefficient of x^n between equation ① and ④, we get $Q = a_0$.

So, eqⁿ ④ becomes

$$f_n(x) \equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) \quad \text{⑤}$$

Therefore, $f_n(x)=0$ has n roots.

Now, we have to prove $f_n(x)=0$ has got n and only n roots.

Let $\delta \neq \alpha_1, \alpha_2, \ldots, \alpha_n$

Then, no factor of $f_n(x)=0$ can vanish as is evident from ⑤ and consequently $f_n(x) \neq 0$ for $x=\delta$.

Hence, $f_n(x)=0$ cannot have more than n roots.

This proves the theorem.